Analytic Existence and Uniqueness Results for PDE-Based Image Reconstruction with the Laplacian

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PDE-based Image Inpainting

(a) Image repair (b) Image compression

Figure: Important inpainting applications

One of the most successful setups uses the Laplacian [Noma & Misulia, 1959]:

$$
\begin{aligned}
-\Delta u &= 0, && \text{on } \Omega \setminus \Omega_K \\
u &= f, && \text{on } \partial\Omega_K \\
\partial_n u &= 0, && \text{on } \partial\Omega \setminus \partial\Omega_K
\end{aligned}
$$

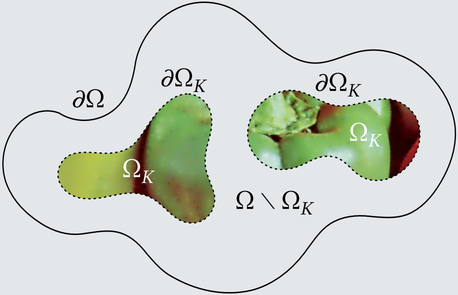

Figure: Generic inpainting model with known data f on Ω_K.

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Frequently Encountered Variations

- Using a mask function c [Weickert & Welk, 2006]:

$$\begin{aligned} c(u-f)+(1-c)(-\Delta u) &= 0, \quad \text{on } \Omega \\ \partial_n u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

with $c\colon\ \Omega \to \{0,1\}$, or $[0,1]$, or even $\mathbb{R}$

- Rewriting as a Helmholtz equation [H., 2017]:

$$\begin{aligned} -\Delta u + \frac{c}{1-c} u &= \frac{c}{c-1} f, && \text{on } \Omega \setminus \Omega_K \\ u &= f, && \text{on } \partial\Omega_K \\ \partial_n u &= 0, && \text{on } \partial\Omega \end{aligned}$$

Motivation and Goals

- Laplace equation is well understood, but:
  - regularity of solutions depends on boundary data
  - impact of c has received little attention
  - boundaries may prevent the existence of a solution
- On the discrete side, we know that:
  - most discretised formulations are well-posed
  - they require upper bounds on c
  - upper bound depends on discretisation of the Laplacian

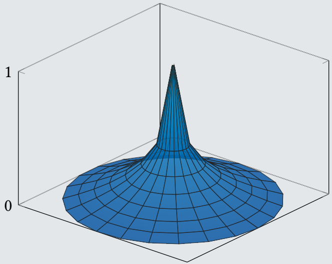
Verdict:

The current situation is not satisfying!

Hidden Pitfalls

$$
\begin{aligned}
-\Delta u &= 0, \quad \text{on } B_1(0) \setminus \{0\} \\
u &= 1, \quad \text{when } x = 0 \\
u &= 0, \quad \text{when } \|x\| = 1
\end{aligned}
$$

- PDE does **not** have a solution
- naive finite difference discretisation yields linear system
- system matrix is **regular** (Geršgorin's disk theorem)



Model Assumptions

The following assumptions are always assumed to hold.

- Ω is open and bounded with C^∞ boundary $\partial\Omega$.
- $f : \Omega \to \mathbb{R}$ is C^∞
- $\Omega_K \subsetneq \Omega$ is closed, has positive measure, and C^∞ boundary
- for mixed boundary value problems, $\partial\Omega \cap \partial\Omega_K = \emptyset$

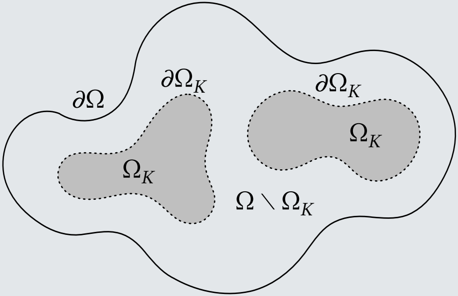
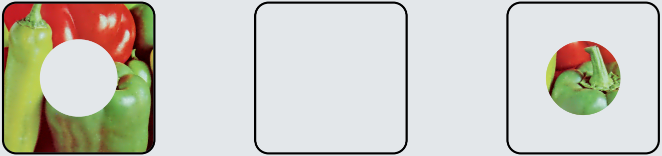


Figure: Assumptions on the considered domain

Possible Setups

Depending on size and shape of Ω_K we obtain

- Dirichlet problem with (in-)homogeneous boundary conditions
- Neumann problem with homogeneous boundary conditions
- mixed Neumann, Dirichlet boundary conditions



(a) Dirichlet problem **(b)** Neumann problem **(c)** Mixed problem

Figure: Possible setups to consider

Theorem ([Ern & Germond, 2004])

The classic inpainting formulation

$$\begin{aligned} -\Delta u &= 0, \quad \text{on } \Omega \setminus \Omega_K \\ u &= f, \quad \text{on } \partial\Omega_K \\ \partial_n u &= 0, \quad \text{on } \partial\Omega \setminus \partial\Omega_K \end{aligned}$$

is well-posed. Solutions live in the Sobolev space H^1.

Well-posedness is understood in the sense of Hadamard:

1. Solution u exists for every possible Ω, Ω_K, and f
2. Solution u is unique
3. Solution u depends continuously on the data

From Binary to Arbitrary Masks c

Theorem ([Cantrell & Cosner, 2003])

Let $0 \leq c < 1$ hold for all $x \in \Omega$. The inpainting formulation

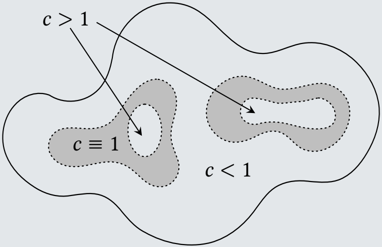
$$\begin{aligned} c\,(u - f) + (1 - c)\,(-\Delta)\,u &= 0, \quad \text{on } \Omega \\ \partial_n u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

is well-posed. Solutions live in the space $C^{2,\alpha}(\overline{\Omega})$.

- $c < 1$ is not really what we want
- $c \geq 1$ allows contrast enhancing and is important for applications

Idea: Separating Regions

- **Continuous** c separates regions where $c > 1$ and $c < 1$



- Regions can be handled independently.
- PDE can be rewritten for each region as:

$$
\begin{aligned}
c\,(u - f) - (1 - c)\,\Delta u &= 0, && \text{on } \Omega \setminus \Omega_K \\
u &= f, && \text{on } \Omega_K \\
\partial_n u &= 0, && \text{on } \partial\Omega \setminus \partial\Omega_K
\end{aligned}
$$

From Laplace to Helmholtz

Whenever $c > 1$ or $c < 1$, the PDE

$$\begin{aligned} c\,(u - f) - (1-c)\,\Delta u = 0, &\quad \text{on } \Omega \setminus \Omega_K \\ u = f, &\quad \text{on } \Omega_K \\ \partial_n u = 0, &\quad \text{on } \partial\Omega \setminus \partial\Omega_K \end{aligned}$$

can be rewritten as an inhomogeneous Helmholtz equation

$$\begin{aligned} -\Delta u + \eta u = g &\quad \text{on } \Omega \setminus \Omega_K \\ u = f &\quad \text{on } \partial\Omega_K \\ \partial_n u = 0 &\quad \text{on } \partial\Omega \setminus \partial\Omega_K \end{aligned}$$

with $g := \eta f$ and non-constant refraction $\eta := \frac{c}{1-c}$

Weak Formulation of the Helmholtz Equation

The corresponding weak formulation reads

$$\int_{\Omega\setminus\Omega_K} \nabla u \nabla \varphi + \eta u \varphi \,\mathrm{d}x = \int_{\Omega\setminus\Omega_K} \Delta f \varphi \,\mathrm{d}x - \int_{\partial\Omega\setminus\partial\Omega_K} \varphi \partial_n f \,\mathrm{d}S$$

which needs to be solved in

$$V := \left\{ \phi \in H^1(\Omega \setminus \Omega_K) \,\middle|\, \phi\Big|_{\partial\Omega_K} \equiv 0 \right\}$$

Note that:

- $\eta > 0$ for $c < 1$: Lax-Milgram is applicable
- $\eta < 0$ for $c > 1$: Lax-Milgram is **not** applicable

Existence of a Weak Solution

We observe:

- the space V can be equipped with the scalar product

$$\langle u, \varphi \rangle := \int\limits_{\Omega \setminus \Omega_K} \nabla u \nabla \varphi \, \mathrm{d}x$$

- the Riesz representation theorem asserts the existence of the bounded linear operator $B : V \to V$

$$\langle Bu, \varphi \rangle := \int\limits_{\Omega \setminus \Omega_K} -\eta u \varphi \, \mathrm{d}x$$

- $H^1 \hookrightarrow L^2$ implies that B is compact and selfadjoint

- Our weak formulation

$$\underbrace{\int_{\Omega\setminus\Omega_K} \nabla u \nabla \varphi \,\mathrm{d}x - \int_{\Omega\setminus\Omega_K} -\eta u \varphi \,\mathrm{d}x}_{=\langle u,\varphi\rangle - \langle Bu,\varphi\rangle} = \underbrace{\int_{\Omega\setminus\Omega_K} \Delta f \varphi \,\mathrm{d}x - \int_{\partial\Omega\setminus\partial\Omega_K} \varphi \partial_n f \,\mathrm{d}S}_{=:\ell(\varphi)}$$

can now be rewritten as a variational equation

$$(I - B)u = \ell$$

- Fredholm alternative implies that $I - B$ is invertible if it is injective
- $I - B$ is injective if $\lambda = 1$ is **not** an eigenvalue of B

- B is a compact operator: its spectrum is countable
- probability of $\lambda = 1$ being an eigenvalue is 0
- the problem is almost certainly well posed

Theorem (Well-posedness of PDE-based inpainting)

Inpainting with continuous c *is almost certainly well posed.*

Take Home Message

- inpainting with the Laplacian is almost always well posed
- solutions are at least in H^1
- implies that discrete equations should be solvable almost always
- non-binary masks are related to the Helmholtz equation

Thank you very much for your attention!

For more information:

https://www.b-tu.de/fg-angewandte-mathematik/