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Matrix-Valued Levelings for Colour Images

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What are levelings?

- scale space representation based on morphological filters
- remove details but preserve contours

Definition (Leveling [Meyer, 1998])

An image g is a **leveling** of an image f iff $\forall(p, q)$ neighbors:

$$g_p > g_q \implies f_p \geqslant g_p \text{ and } g_q \geqslant f_q$$

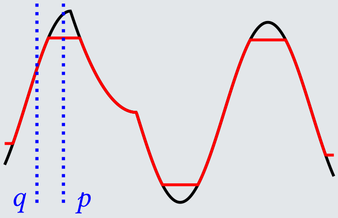


Figure: function f (black) and its leveling g (red)

Properties of levelings

- invariances: translations, rotations, illumination changes
- contours in levelings g have stronger contours in function f
- scale space property:

  Image gets simplified, but contours remain localized!

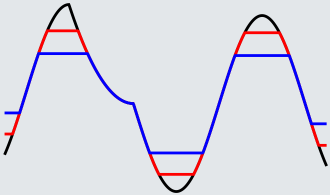


Figure: function f (black) and its levelings g_1 and g_2 (red and blue)

- **but:** levelings require an ordering relation!

Our Motivation:

- applications: texture processing, image compression, ...
- we want to work with colour valued images

Our Goals:

- provide alternative to vectorial levelings [Meyer, 2000]
- extension to colour images
- investigate properties of levelings on colour images

Our Strategy:

- consider matrix valued colour space with partial ordering

Algorithmic Aspects

- existence of discrete iterative routine
- existence of a PDE-based formulation
- unclear which one is better suited for our tasks
- in the following we denote:

$$(f_1 \vee f_2)(x) := \sup\{ f_1(x),\ f_2(x)\}$$
$$(f_1 \wedge f_2)(x) := \inf\{ f_1(x),\ f_2(x)\}$$
$$\delta_B(f)(x) := \sup_{a \in B}\{ f(x-a)\}$$
$$\varepsilon_B(f)(x) := \inf_{a \in B}\{ f(x+a)\}$$

Discrete formulation

We need:

- grey-valued **input image** f
- **marker image** M
- **structuring element** B

Definition (Discrete Levelings [Meyer, 1998])

A **leveling** is a fixed-point of

$$u_{k+1} = [f \wedge \delta_B(u_k)] \vee \varepsilon_B(u_k), \quad \text{with} \quad u_0 = M$$

PDE-based formulation

- let Ω be our image domain
- PDE-based dilation and erosion are given by:

$$\partial_t u = \pm \|\nabla u\|, \qquad \forall x \in \Omega,\ \forall t > 0$$

Definition (PDE-based levelings [Maragos & Meyer, 1999])

$$\begin{aligned} \partial_t u &= \operatorname{sgn}(u - f)\, \|\nabla u\|, && \forall x \in \Omega \\ u(0,x) &= (K_\sigma * f)(x), && \forall x \in \Omega \\ \partial_n u(t,x) &= 0, && \forall x \in \partial\Omega,\ \forall t \geq 0 \end{aligned}$$

Comparison of the models

- sign of $u_k - f$ switches between dilation and erosion in

$$[f \wedge \delta_B(u_k)] \vee \varepsilon_B(u_k)$$

- $\operatorname{sgn}(u - f)$ switches between dilation and erosion in

$$\partial_t u = \operatorname{sgn}(u - f)\,\|\nabla u\|$$

- number of iterations behaves like stopping time

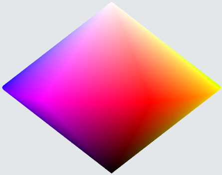
Both formulations show similar behaviour!

Bi-cone shaped colour space [Burgeth & Kleefeld, 2013]

- Bi-cone colour space has values in $\mathbb{S}^2_+$
- Loewner order gives partial ordering in $\mathbb{S}^2_+$

$$A \succcurlyeq B \Leftrightarrow A - B \in \mathbb{S}^2_+$$

- partial ordering allows definition of supremum and infimum
- sup and inf allow definition of dilation and erosion



Discrete colour morphology

- naive approach: compute levelings for each channel separately
  - simple and fast
  - decoupling the channels can lead to wrong results
  - unknown optimal structuring element
- use bi-cone shaped colour space
  - not all colours are comparable
  - arithmetics in this space are not trivial
  - unknown optimal structuring element

PDE-based colour-valued levelings

$$\partial_t u = \operatorname{sgn}(u - f)\|\nabla u\|$$

- finite difference forward time discretisation
- Rouy-Tourin discretisation for derivatives

$$u_z \approx \max\big\{\max\{D_z^- u,\ 0\},\ -\min\big\{D_z^+ u,\ 0\big\}\big\}$$

with $D_z^\pm$ being forward/backward difference along z

- max / min computation of matrices A and B by

$$\frac{1}{2}\left(A + B \pm |A - B|\right)$$

Several possibilities for the sign function:

- based on Loewner order

$$\operatorname{sgn}(u-f)=\begin{cases}+1, & u \succcurlyeq f\\ -1, & u \preccurlyeq f\\ 0, & \text{else}\end{cases}$$

- apply sgn on eigenvalues and use Jordan product

$$A \bullet B := \frac{1}{2}(AB+BA)$$

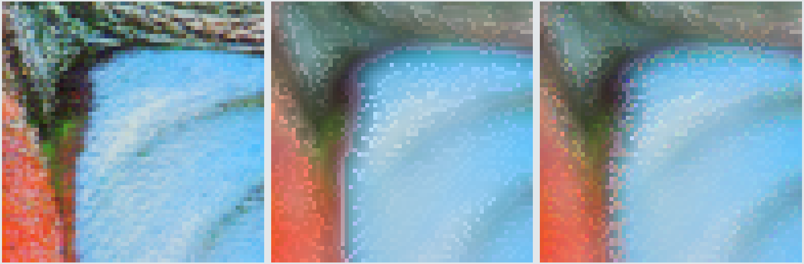
for the matrix-matrix product $\operatorname{sgn}(u-f)\bullet\|\nabla u\|$

Discrete levelings

(a) input (b) channelwise (c) Loewner

Figure: channelwise approach causes false colours

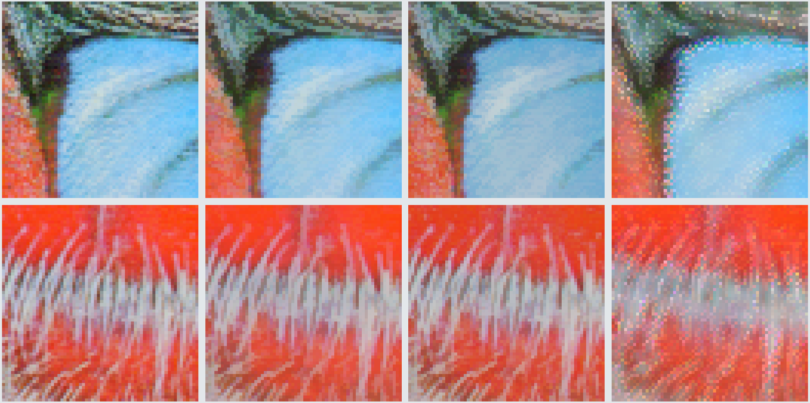
PDE-based model



(a) original **(b)** Loewner order **(c)** Jordan product

- small scale structures are removed
- Loewner ordering yields clearer edges

Texture discrimination

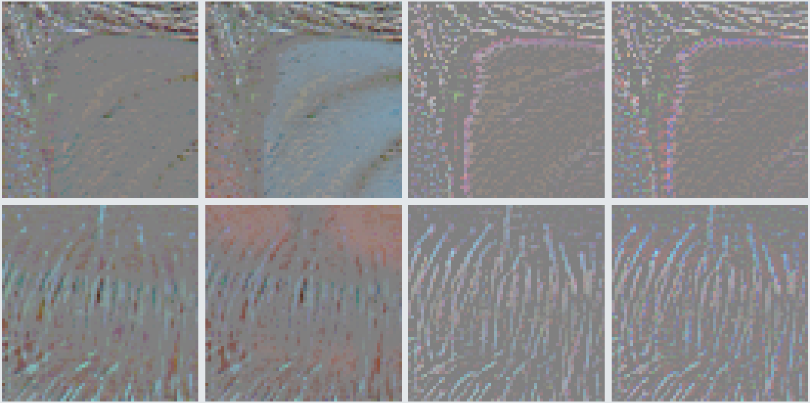


(a) channelwise **(b)** discrete **(c)** PDE-Loewner **(d)** PDE-Jordan

Experimental Results (4)

Texture discrimination

difference to original image reveals texture information



(a) channelwise **(b)** discrete **(c)** PDE-Loewner **(d)** PDE-Jordan

Conclusions

- The discrete approach is fast and simple
  - channelwise computations may yield false colours
  - filters texture and structure information
- PDE-based models are numerically challenging
  - Loewner ordering has fewer artifacts
  - better textures filtering

Thank you very much for your attention!

For more information:

`https://www.b-tu.de/fg-angewandte-mathematik/`