Continuous Spatial and Tonal Point Optimisation for Interpolation and Approximation of Convex Signals with Homogeneous Diffusion

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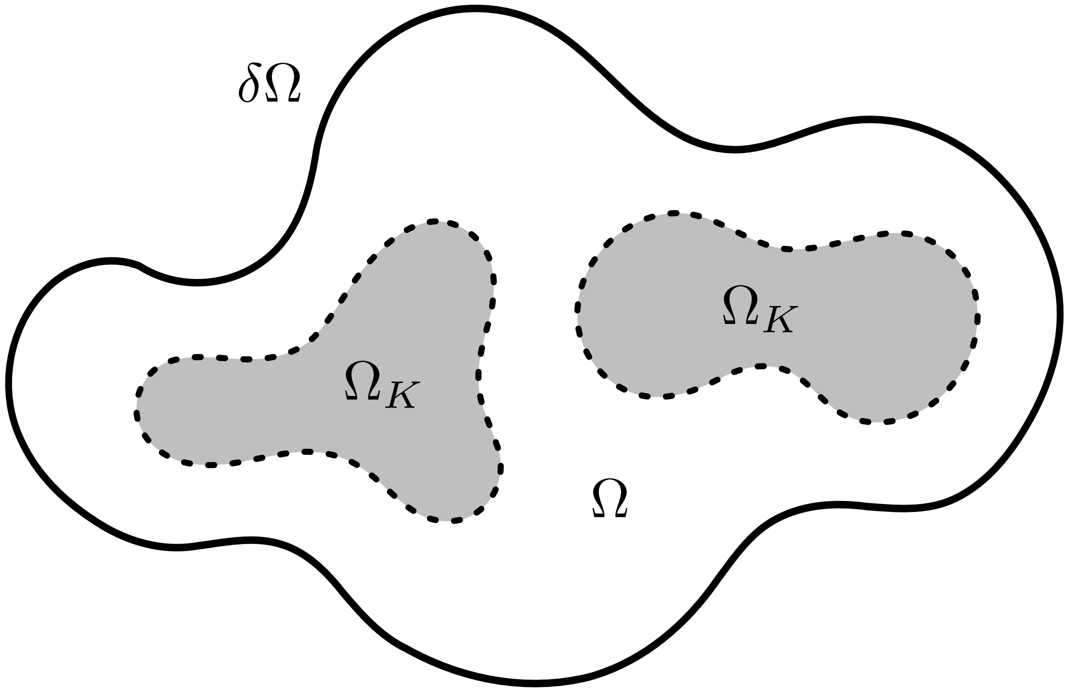
Inpainting with homogeneous diffusion

Consider the Laplace equation with mixed boundary conditions.



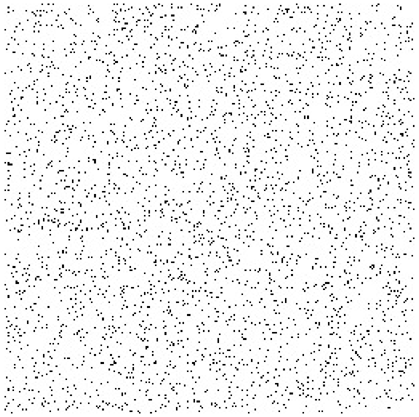
$$\begin{cases} \Delta u = 0, & \text{on } \Omega \\ u = g, & \text{on } \Omega_K \\ \partial_n u = 0, & \text{on } \delta\Omega \end{cases}$$

- Ω_K represents known data.
- $\Omega \setminus \Omega_K$ region to be inpaintend.
- Image reconstructions given by solution u of boundary value problem.

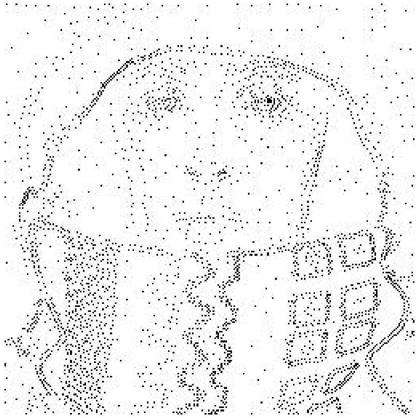
Influence of the inpainting data

Choice of Ω_K has tremendous impact on the reconstruction.

Original Image



Badly chosen Ω_K.



Well chosen Ω_K.

Bad reconstruction.

Good reconstruction.

How to optimise the interpolation data Ω_K?

Outlook

- **Optimisation in an interpolation framework**
  - Problem formulation
  - A new algorithm for optimal interpolation data
  - Theoretical analysis
  - Example
- **Optimisation in an approximation framework**
  - Problem formulation
  - An algorithm for optimal approximation data
  - Theoretical results
  - Example
- **Conclusions**

Analysis in the 1D setting

Simplification: Only consider 1D strictly convex functions $f : [a, b] \to \mathbb{R}$.

Advantages:

- Inpainting simplifies to piecewise linear spline interpolation.
- Analytic expression for reconstruction is available.
  We write (assuming $c_0 := a$ and $c_N := b$):

$$\ell^f\left(x; c_0, \ldots, c_N\right) := \sum_{i=0}^{N-1} \left(\frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}(x - c_i) + f(c_i) \right) \chi_{[c_i, c_{i+1}]}(x)$$

  for the linear spline interpolating f at positions c_0, c_1, ..., c_N.
  $\chi_M(x)$ being the indicator function of the set M.
- Interpolation error (L_1 sense) on the interval $[a, b]$ given by

$$E\left(\{c_i\}_{i=0}^{N}, f\right) := \int_a^b \left|\ell^f\left(x; c_0, \ldots, c_N\right) - f(x)\right| \mathrm{d}x$$

Problem formulation

Task:

Find c_0, …, c_N that minimise interpolation error $E\left(\{c_i\}_{i=0}^N, f\right)$.

Observe:

- Error simplifies to

$$E\left(\{c_i\}_{i=0}^N, f\right) = \frac{1}{2}\sum_{i=0}^{N-1} \left(c_{i+1} - c_i\right)\left(f\left(c_{i+1}\right) + f\left(c_i\right)\right) - \int_a^b f(x)\,\mathrm{d}x$$

- Necessary conditions for a minimum:

$$f'\left(c_i\right) = \frac{f\left(c_{i+1}\right) - f\left(c_{i-1}\right)}{c_{i+1} - c_{i-1}}, \quad \forall i = 1, \ldots, N-1$$

Necessary optimality conditions

- $E\left(\{c_i\}_{i=0}^{N}, f\right)$ is convex for $N = 2$ and generally non-convex for $N > 2$.
- The requirement

$$f'\left(c_i\right) = \frac{f\left(c_{i+1}\right) - f\left(c_{i-1}\right)}{c_{i+1} - c_{i-1}}, \quad \forall i = 1, \ldots, N-1$$

  is a necessary condition, but *not* sufficient.
- Optimal c_i depends only on direct neighbors c_{i-1} and c_{i+1}.

A new algorithm for optimal knots

Algorithm:

1. Start with arbitrary knot distribution $\{c_i^0\}_{i=0}^N$.
2. Update alternatively

$$c_{2i}^{k+1} = (f')^{-1}\left(\frac{f\left(c_{2i+1}^k\right) - f\left(c_{2i-1}^k\right)}{c_{2i+1}^k - c_{2i-1}^k}\right) \quad \forall i$$

$$c_{2i+1}^{k+1} = (f')^{-1}\left(\frac{f\left(c_{2i+2}^k\right) - f\left(c_{2i}^k\right)}{c_{2i+2}^k - c_{2i}^k}\right) \quad \forall i$$

until a fixpoint is reached.

Theoretical properties of the algorithm

One can show:

1. Order of knots is preserved during iteration.

$$c_{i-1}^k < c_i^k < c_{i+1}^k \Rightarrow c_{i-1}^{k+1} < c_i^{k+1} < c_{i+1}^{k+1} \quad \forall i$$

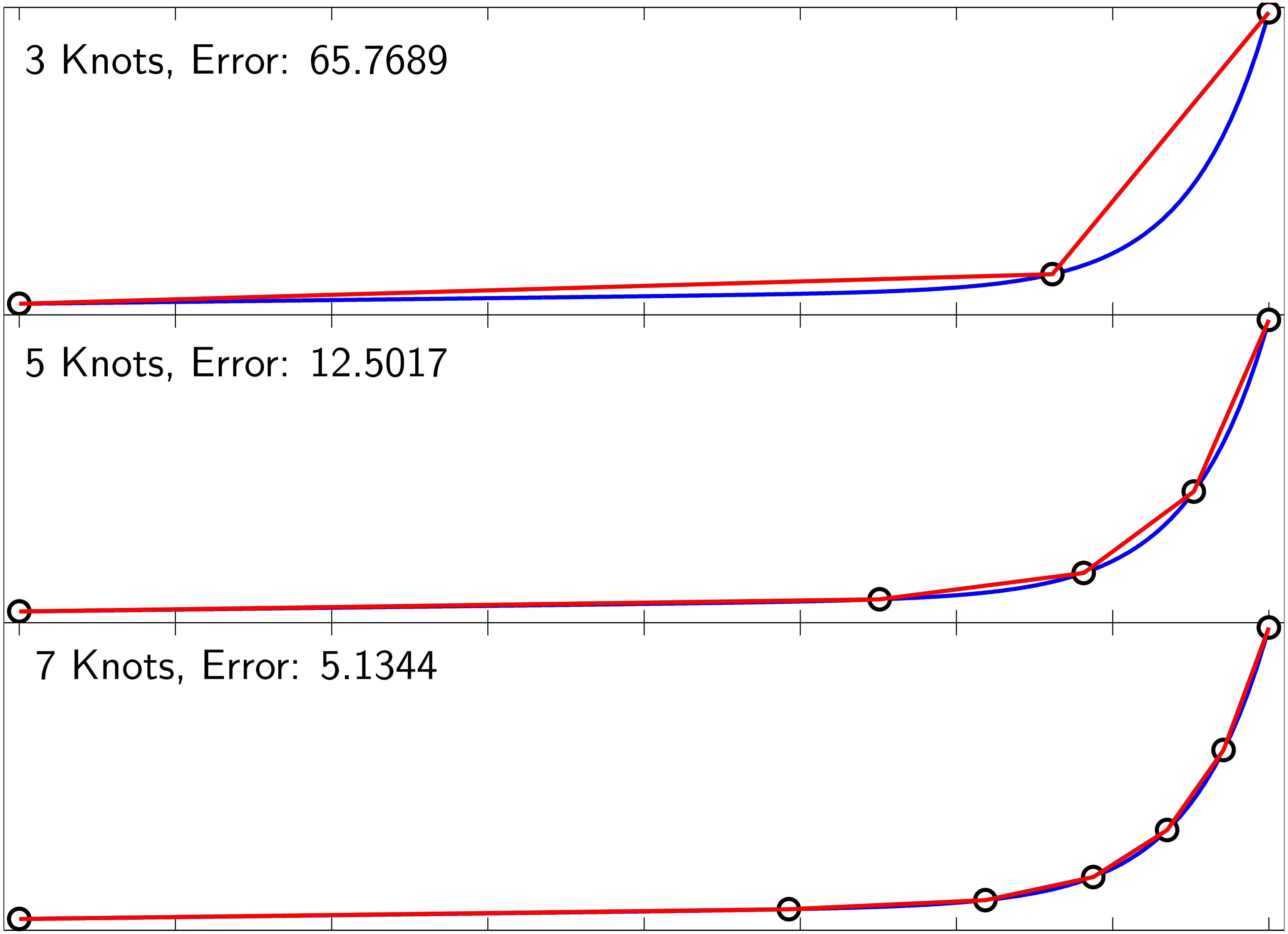
2. Sequence $\left(E\left(\{c_i^k\}_{i=0}^N, f\right)\right)_k$ ist monotonically decreasing.

$$E\left(\{c_i^0\}_{i=0}^N, f\right) \geqslant \ldots \geqslant E\left(\{c_i^k\}_{i=0}^N, f\right) \geqslant E\left(\{c_i^{k+1}\}_{i=0}^N, f\right) \geqslant \ldots$$

3. Sequence $\left(E\left(\{c_i^k\}_{i=0}^N, f\right)\right)_k$ is convergent.
4. Sequence $\left(\{c_i^k\}_{i=0}^N\right)_k$ contains a convergent subsequence.

Numerical example

Consider $f(x) = \exp(2x - 3) + x$ on the interval $[-4, 4]$ with 3, 5 and 7 knots.



From interpolation to approximation

- So far only optimisation of the spatial location c_i.
- *[Mainberger et al., 2011]* optimised c_i and $f(c_i)$ *separately*.

  $\Rightarrow$ Resulted in large quality gains.

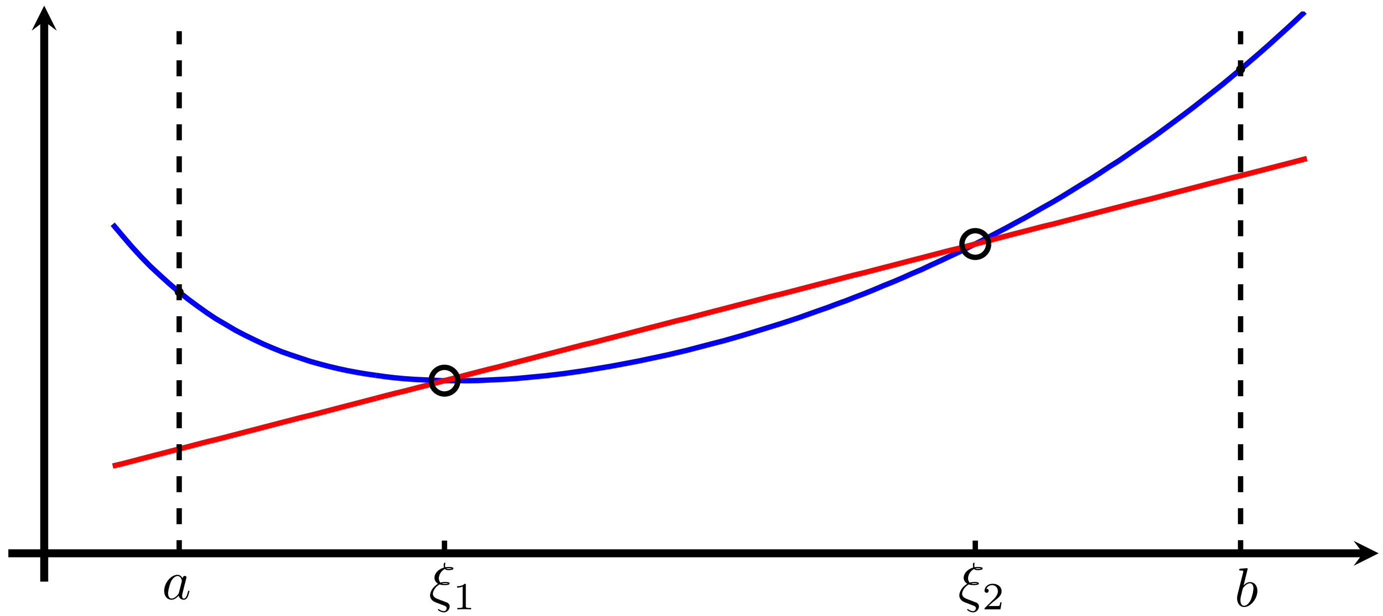
Can we optimise c_i and $f(c_i)$ *simultaneously*?

- Requires abandoning interpolation and using approximation methods.

Optimal linear approximation of strictly convex functions

On $[a,b]$, the optimal line *approximating* a *strictly convex* function f interpolates at

$$\xi_1 := \frac{3}{4}a + \frac{1}{4}b \quad \text{and} \quad \xi_2 := \frac{1}{4}a + \frac{3}{4}b$$



Proof: *[Rice, 1964]*.

Optimal piecewise linear approximation of strictly convex functions

In *[Hamideh, 2002]*, the author suggested the following algorithm.

Algorithm:

1. Start with arbitrary knot distribution $\{c_i^0\}_{i=0}^N$.
2. On each intervall $\left[c_i^k, c_{i+1}^k\right]$, compute optimal line $\ell_i(x)$ interpolating f at

$$\xi_{i,1} := \frac{3}{4}c_i^k + \frac{1}{4}c_{i+1}^k \quad \text{and} \quad \xi_{i,2} := \frac{1}{4}c_i^k + \frac{3}{4}c_{i+1}^k$$

$$\ell_i\left(x\right) := \frac{f\left(\xi_{i,2}\right) - f\left(\xi_{i,1}\right)}{\xi_{i,2} - \xi_{i,1}}\left(x - \xi_{i,1}\right) + f\left(\xi_{i,1}\right)$$

3. Set new c_i^{k+1} at the intersection point between $\ell_i(x)$ and $\ell_{i+1}(x)$. E.g. solve

$$\ell_i(c_i^{k+1}) = \ell_{i+1}(c_i^{k+1})$$

4. Repeat until convergence is reached.

Theoretical properties

One can show:

1. The sequence of approximation errors is convergent.
   Proof: *[Hamideh, 2002]*.
2. For all $i = 1, \ldots, N-1$ we have

$$\lim_{k\to\infty} \inf \left|c_{i+1}^k - c_i^k\right| > 0 \quad \text{and} \quad \lim_{k\to\infty} \left|c_i^{k+1} - c_i^k\right| = 0$$

   Proof: *[Hamideh, 2002]*.
3. Convergence towards a optimal solution can be proven under some additional assumptions.
   Proof: *[Hamideh, 2002]*.

Theoretical properties

One can show:

4. The knots c_i are optimal if they solve the *continuity condition*:

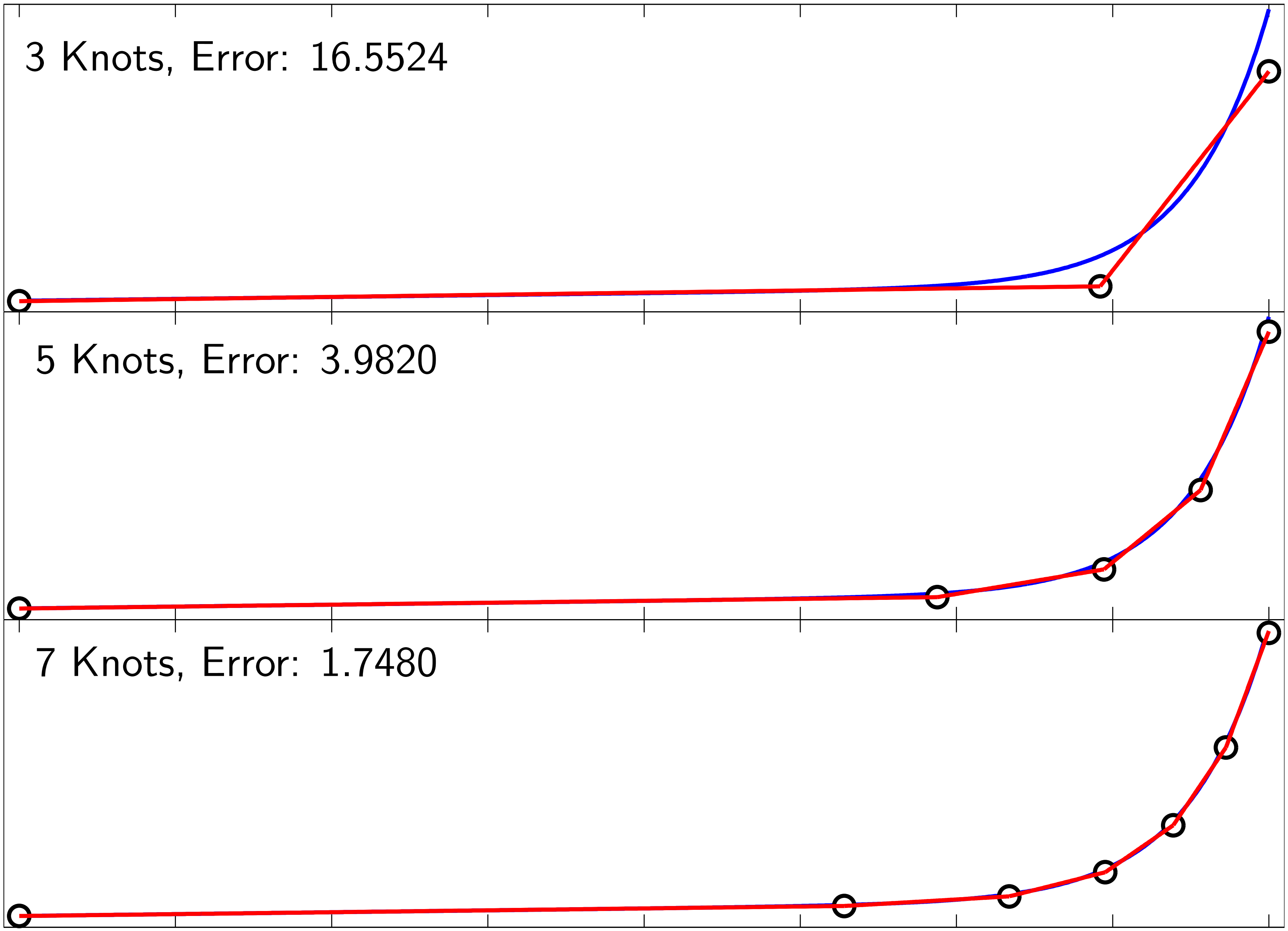
$$f\left(\frac{3c_{i-1}+c_i}{4}\right) - 3f\left(\frac{c_{i-1}+3c_i}{4}\right) + 3f\left(\frac{3c_i+c_{i+1}}{4}\right) - f\left(\frac{c_i+3c_{i+1}}{4}\right) = 0$$

   for all i.
   Proof: *[Kioustelidis and Spyropoulos, 1978]*

5. The algorithm of Hamideh corresponds to an inexact Newton method to solve the continuity conditions.
   Proof: *[Chieppa, 2009]*.

Numerical example

Consider $f(x) = \exp(2x-3) + x$ on the interval $[-4, 4]$ with 3, 5 and 7 knots.



Interpolation vs. approximation

The approximation framework reduces the error significantly.

Number of Knots	L_1 Error	
	Interpolation	Approximation
3	65.7689	16.5524
5	12.5017	3.9820
7	5.1344	1.7480

Note: Both approaches have similar complexity and runtimes.

Combined spatial and tonal optimisation is possible and pays off!

Summary and conclusions

We have seen:

- Two strategies to optimise interpolation data in 1D.
- Approximation frameworks can outperform pure interpolation approaches.

Potential Issues:

- Applications to 2D images cumbersome.

  $\Rightarrow$ Apply alternatively along every dimension.
- Convexity requirement is essential and a severe restriction.

  $\Rightarrow$ Segmentation into convex/concave regions becomes necessary.

Ongoing Work:

- Improved handling of 2D image data.
- Extensions to other interpolation methods.

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Thank you very much for your attention.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20